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Energy decay estimates for the Bernoulli–Euler-type equation with a local degenerate dissipation

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ABSTRACT

In this paper, we study the decay property of the solutions to the Bernoulli–Euler-type equation with a local degenerate dissipation.

1. Introduction

In this paper, we consider the decay property of the solutions to the Bernoulli–Euler-type equation with a local degenerate dissipation

$$u_{tt} + \Delta^2 u - \alpha \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right) \Delta u + a(x) u_t = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

$$u = \frac{\partial u}{\partial \eta} = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{1.2}$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \quad \text{in } \Omega, \tag{1.3}$$

where Ω is a bounded domain in R^N with smooth boundary Γ, α is a positive constant and $a(x)$ is a nonnegative smooth function on $\bar{\Omega}$ which may vanish somewhere in $\bar{\Omega}$. It is interesting to observe that when $N = 1$ the model considered here is a general mathematical formulation of a problem arising in the dynamic buckling of a hinged extensible beam under an axial force.

An energy decay property for the problem (1.1)–(1.3) means that there exists a function $g(t)$ such that

$$E(t) \le g(t) E(0), \quad t \ge 0 \text{ with } \lim_{t \to \infty} g(t) = 0,$$

where $E(t)$ is the energy of the solution at time t to the problem (1.1)–(1.3), defined by

$$E(t) = \frac{1}{2} \int_\Omega \left(u_t^2 + |\Delta u|^2 \right) \mathrm{d}x + \frac{\alpha}{4} \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^2. \tag{1.4}$$

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It is well known that the decay rates of the energy $E(t)$ depend on the geometrical shape, the dimension of the domain and the coefficient function $a(x)$ in the dissipative term. This problem for the following wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0 \quad \text{in } \Omega \times (0, \infty), \\ u = 0 \quad \text{on } \Gamma \times (0, \infty), \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \quad \text{in } \Omega \end{cases} \tag{1.5}$$

has been studied by many authors. We refer the reader to [1–13].

When $a(x) \geq \epsilon_0 > 0$ on $\bar{\Omega}$, it is easy to show that the energy of solutions to the problem (1.5) decays exponentially to 0 as $t \to \infty$; that is, there exists a constant $\lambda > 0$ such that

$$E(t) = \mathcal{O}(e^{-\lambda t}) \tag{1.6}$$

for any initial condition $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ (see Rauch and Taylor [10]). Dafermos [4] and Haraux [5] showed that the energy of all solutions to the problem (1.5) goes to 0 as $t \to \infty$; that is,

$$\lim_{t\to\infty} E(t) = 0$$

if $a(x_0) > 0$ for some $x_0 \in \Omega$. Zuazua [13] proved that estimate (1.6) holds for all solutions to the problem (1.5) if $a(x) \geq \epsilon_0 > 0$ only on a neighborhood ω in $\bar{\Omega}$ of $\Gamma(x_0)$ (see Lions [14]), which is a part of the boundary given by

$$\Gamma(x_0) = \{x \in \partial\Omega \,|\, (x - x_0) \cdot \nu(x) \geq 0\},$$

where $x_0 \in \mathbb{R}^N$ is arbitrarily fixed and $\nu(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$. Nakao [8] considered containing the case of a degenerate dissipation as follows. Let $x_0 \in \mathbb{R}^N$ and ω be a neighborhood in $\bar{\Omega}$ of $\Gamma(x_0)$. Assume that

$$a(x) > 0 \quad \text{a.e. } x \in \omega \quad \text{and} \quad \int_\omega a(x)^{-p} \mathrm{d}x < \infty \tag{1.7}$$

for some $p \in (0, 1)$. Then the energy decays polynomially; that is,

$$E(t) = \mathcal{O}\left(t^{-\frac{2mp}{N}}\right), \tag{1.8}$$

where $a(\cdot) \in C^{m-1}(\bar{\Omega})$ and $(u_0, u_1) \in H^{m+1}(\Omega) \times H^m(\Omega)$ satisfies the compatibility condition of the m-th order relative to the problem (1.5). Later, under the same hypotheses on the data, Tcheugoué Tébou [11] proved that, for any given $N \geq 1$, if $N < 2m$ and $0 < p < \infty$, then the energy $E(t)$ satisfies (1.8), and furthermore, when $N \geq 2m$ and $p > 0$, satisfying $N - 2m \leq mp$, the energy $E(t)$ satisfies

$$E(t) = \mathcal{O}\left(t^{-\frac{mp}{N}}\right). \tag{1.9}$$

Kang et al. [6] proved sharp energy decay estimates for the wave equation with a local degenerate dissipation. Cavalcanti et al. [3] studied the asymptotic stability of the wave equation on compact manifolds and locally distributed damping.

The same problems have also been addressed for the Bernoulli–Euler equation. We refer the reader to [15–19]. Cavalcanti et al. [16] considered the asymptotic stability for the nonlinear and generalized damped extensible plate equation. Recently, Charão et al. [17] proved the polynomial decay for the energy of solutions of a nonlinear plate equation of Bernoulli–Euler type with a nonlinear localized damping term.

Motivated by work in [6,9,17], we obtain in this paper precise energy decay estimates for the Bernoulli–Euler-type equation with a local degenerate dissipation.

2. Statement of the main result

Throughout this paper we shall use familiar Sobolev spaces $H^m(\Omega)$ with respect to the norm

$$\|f\|_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^2(\Omega)},$$

where $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1}, \dots, \partial x_N{}^{\alpha_N})$, $\alpha = (\alpha_1, \dots, \alpha_N)$, $\alpha_i \geq 0$, $i = 1, \dots, N$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$ and $\|\cdot\|_{L^p(\Omega)}$ denotes the L^p norm on an open set $\Omega \in \mathbb{R}^N$, $1 \leq p \leq \infty$. We begin with the following well-known lemmas, which will be used in the main result.

Lemma 2.1 (*Gagliardo–Nirenberg Inequality*)**.** *Let $1 \leq r < p \leq \infty$, $1 \leq q \leq p$ and $0 \leq k \leq m$. Then we have the inequality*

$$\|v\|_{W^{k,p}(\Omega)} \leq C\|v\|^\theta_{W^{m,q}(\Omega)} \|v\|^{1-\theta}_{L^r(\Omega)} \quad \text{for } v \in W^{m,q}(\Omega) \cap L^r(\Omega)$$

with some $C > 0$ and

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{p}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{q}\right)^{-1}$$

provided that $0 < \theta \leq 1$($0 < \theta < 1$ if $p = \infty$, mq = integer).

We also need the following lemma of Nakao [8].

Lemma 2.2. *Let $\phi(t)$ be a nonnegative function on $[0, \infty)$ satisfying*

$$\sup_{t \le s \le t+T} \phi(s)^{1+\gamma} \le C_\phi \{\phi(t) - \phi(t+T)\},$$

where T, γ and C_ϕ are some positive constants. Then $\phi(t)$ has the decay property

$$\phi(t) \le \left\{ \phi(0)^{-\gamma} + \frac{\gamma}{C_\phi}(t-T)^+ \right\}^{-\frac{1}{\gamma}}$$

for all $t \ge 0$, where the notation α^+ signifies $\max\{\alpha, 0\}$.

The existence and the regularity of the solution u of the problem (1.1)–(1.3) are given by the following well-known result.

Theorem 2.1. *Let $a(\cdot) \in L^\infty(\Omega)$. Then, for each $T > 0$, we have the following.*

1. *If $u_0 \in H^4(\Omega) \cap H_0^2(\Omega)$ and $u_1 \in H_0^2(\Omega)$, then system (1.1)–(1.3) admits a unique solution having the regularity*
$$u \in C([0, T]; H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0, T]; H_0^2(\Omega)).$$
2. *If $u_0 \in H_0^2(\Omega)$ and $u_1 \in L^2(\Omega)$, then system (1.1)–(1.3) admits a unique solution having the regularity*
$$u \in C([0, T]; H_0^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)).$$

The existence of both strong and weak solutions may be proven either by the theory of a semigroup or by the Galerkin method. The main result of this paper reads as follows.

Theorem 2.2. *Let $(u_0, u_1) \in H^4(\Omega) \cap H_0^2(\Omega) \times H_0^2(\Omega)$. Suppose that $a(x) \ge 0$ on $\bar{\Omega}$, $a(\cdot)$ belongs to $C^1(\bar{\Omega})$ and there exist $x_0 \in \mathbb{R}^N$ and ω, the intersection of $\bar{\Omega}$ and a neighborhood of $\Gamma(x_0)$ such that*

$$a(x) > 0 \quad a.e.\ x \in \omega, \qquad \int_\omega a(x)^{-p} \mathrm{d}x < \infty,$$

where the parameter $p > 0$ satisfies

$$\begin{cases} 0 < p < \infty & \text{if } 1 \le N < 4, \\ N - 4 \le 2p & \text{if } 4 \le N. \end{cases} \tag{2.1}$$

Then, for any given N, we have the following decay estimates. If $N < 4$, then the energy $E(t)$ satisfies

$$E(t) \le \left(E(0)^{-\frac{N}{4p}} + \frac{N}{4p} C(u_0, u_1, T)^{-\left(1+\frac{N}{4p}\right)} (t-T)^+ \right)^{-\frac{4p}{N}} \tag{2.2}$$

for all $t \ge 0$, where $C(u_0, u_1, T)$ is a positive constant depending on the initial data u_0, u_1 and T.

On the other hand, if $N \ge 4$, then the energy $E(t)$ satisfies

$$E(t) \le \left(E(0)^{-\frac{N}{4p}} + \frac{N}{4p} \tilde{C}(u_0, u_1, T)^{-\left(1+\frac{N}{4p}\right)} (t-T)^+ \right)^{-\frac{4p}{N}} \tag{2.3}$$

for all $t \ge 0$, where $\tilde{C}(u_0, u_1, T)$ is a positive constant depending on the initial data u_0, u_1 and T.

Since $a(x)$ may vanish for $x \in \omega^c$ and may degenerate on $N-1$ submanifolds in ω, our dissipation $a(x)u_t$ is localized at a neighborhood of a part of $\Gamma(x_0)$ as well as degenerate on such a neighborhood.

Remark 2.1. Theorem 2.2 says that if, for any given dimension $N(\ge 1)$, there exists a constant $p > 0$ satisfying (2.1), then we have the decay property

$$E(t) = \mathcal{O}\left(t^{-\frac{4p}{N}}\right)$$

for the solution $u(t) \in C([0, T]; H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0, T]; H_0^2(\Omega))$.

3. Proof of Theorem 2.2

Lemma 3.1. *Let $\eta(x) \in W^{1,\infty}(\Omega)$ and $h(x) = (h_1(x), h_2(x), \dots, h_N(x)) \in (W^{1,\infty}(\Omega))^N$. Then, for the solution $u(t) \in C([0,T]; H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0,T]; H_0^2(\Omega))$ and $T > 0$, we have the following identities:*

$$E(t+T) + \int_t^{t+T} \int_\Omega a(x)|u_t|^2 \mathrm{d}x\mathrm{d}s = E(t);$$

$$\int_t^{t+T} \int_\Omega \eta(x)(|\Delta u|^2 - |u_t|^2 + \alpha\|\nabla u\|^2|\nabla u|^2)\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega \eta(x)a(x)u_t u \mathrm{d}x\mathrm{d}s$$
$$= -(\eta(x)u, u_t)\,|_t^{t+T} - \int_t^{t+T}\int_\Omega [u\Delta u\Delta\eta(x) + 2\Delta u(\nabla u \cdot \nabla\eta(x))]\mathrm{d}x\mathrm{d}s - \alpha\int_t^{t+T} \|\nabla u\|^2 \int_\Omega u(\nabla u\cdot\nabla\eta(x))\mathrm{d}x\mathrm{d}s;$$

$$\frac{1}{2}\int_t^{t+T}\int_\Gamma h(x)\cdot\nu(x)|\Delta u|^2 d\Gamma \mathrm{d}s - (u_t, h(x)\cdot\nabla u)\,|_t^{t+T}$$
$$= \frac{1}{2}\int_t^{t+T}\int_\Omega divh(x)(|u_t|^2 - |\Delta u|^2 - \alpha\|\nabla u\|^2|\nabla u|^2)\mathrm{d}x\mathrm{d}s$$
$$+\alpha\int_t^{t+T}\int_\Omega \|\nabla u\|^2 \sum_{i,j}\frac{\partial h_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega (\Delta h(x)\nabla u)\Delta u\mathrm{d}x\mathrm{d}s$$
$$+2\int_t^{t+T}\int_\Omega \sum_{i,j}\frac{\partial h_j}{\partial x_i}\frac{\partial^2 u}{\partial x_i \partial x_j}\Delta u\mathrm{d}x\mathrm{d}s + \int_t^{t+T}\int_\Omega a(x)h(x)\cdot\nabla u u_t \mathrm{d}x\mathrm{d}s.$$

These identities are derived by multiplying Eq. (1.1) by u_t, $\eta(x)u$ and $h(x)\cdot\nabla u$, respectively, and integrating over $[t, t+T]\times\Omega$, $T > 0$, using integration by parts. Now let us set

$$D(t)^2 \equiv E(t) - E(t+T) = \int_t^{t+T}\int_\Omega a(x)|u_t|^2\mathrm{d}x\mathrm{d}s. \tag{3.1}$$

Using the identities in Lemma 3.1 and proceeding similarly as in the proof of the main theorem of Nakao [8] and Charão et al. [17], we can derive the following inequality, which is a useful inequality for deriving the decay rate of the energy.

Proposition 3.1. *Let $T > 0$ be a sufficiently large number. For the solution $u(t) \in C([0,T]; H^4(\Omega) \cap H_0^2(\Omega)) \cap C^1([0,T]; H_0^2(\Omega))$, there exists a constant $C > 0$ independent of the initial data, such that the estimate*

$$E(t+T) \le C\left\{D(t)^2 + \int_t^{t+T}\int_{\tilde\omega} |u_t|^2\mathrm{d}x\mathrm{d}s\right\}, \tag{3.2}$$

where we set $\tilde\omega = \omega \cap \bar\Omega$.

We are now in a position to prove the main result. In order to complete the proof of Theorem 2.2, we must to find a proper estimate of the term of the integral in the right-hand side of (3.2). We distinguish two cases, corresponding to the hypothesis on N.

Case (i): $N < 4$. By the assumption on $a(x)$ and Lemma 2.1, we obtain

$$\begin{aligned}\int_t^{t+T}\int_{\tilde\omega}|u_t|^2\mathrm{d}x\mathrm{d}s &= \int_t^{t+T}\int_{\tilde\omega} a(x)^{\frac{p}{p+1}}|u_t|^{\frac{2p}{p+1}}a(x)^{\frac{-p}{p+1}}|u_t|^{\frac{2}{p+1}}\mathrm{d}x\mathrm{d}s \\ &\le \left\{\int_t^{t+T}\int_{\tilde\omega} a(x)|u_t|^2\mathrm{d}x\mathrm{d}s\right\}^{\frac{p}{p+1}}\left\{\int_t^{t+T}\int_{\tilde\omega}a(x)^{-p}|u_t|^2\mathrm{d}x\mathrm{d}s\right\}^{\frac{1}{p+1}} \\ &\le D(t)^{\frac{2p}{p+1}}A_p^{\frac{1}{p+1}}T^{\frac{1}{p+1}}\sup_{t\le s\le t+T}\|u_t(s)\|_{L^\infty(\Omega)}^{\frac{2}{p+1}} \\ &\le C_1 D(t)^{\frac{2p}{p+1}}A_p^{\frac{1}{p+1}}T^{\frac{1}{p+1}}\sup_{t\le s\le t+T}\|u_t(s)\|_{H^2(\Omega)}^{\frac{N}{2(p+1)}}\|u_t(s)\|_{L^2(\Omega)}^{\frac{4-N}{2(p+1)}} \\ &\le \tilde C_1 D(t)^{\frac{2p}{p+1}}A_p^{\frac{1}{p+1}}T^{\frac{1}{p+1}}I^{\frac{N}{2(p+1)}}E(t)^{\frac{4-N}{4(p+1)}} \equiv A(t)^2,\end{aligned} \tag{3.3}$$

where $A_p = \int_\omega a(x)^{-p} dx < \infty$ and $C_1, \tilde{C}_1 > 0$ are constants independent of the initial data, and we set $I = c(\|u_0\|^2_{H^4(\Omega)} + \|u_1\|^2_{H^2(\Omega)})^{1/2}$. Thus we have from identity (3.1), Proposition 3.1, (3.3) and Young's inequality on $CA(t)^2$

$$\begin{aligned} E(t) &= D(t)^2 + E(t+T) \\ &\le (C+1)D(t)^2 + CA(t)^2 \\ &\le (C+1)D(t)^2 + \frac{4p+N}{4(p+1)}(C\tilde{C}_1)^{\frac{4(p+1)}{4p+N}}(A_pT)^{\frac{4}{4p+N}} I^{\frac{2N}{4p+N}} D(t)^{\frac{8p}{4p+N}} + \frac{4-N}{4(p+1)}E(t). \end{aligned} \tag{3.4}$$

Using the definition of $D(t)^2$ and (3.4), we get the inequality

$$E(t)^{1+\frac{N}{4p}} \le C(u_0, u_1, T)^{1+\frac{N}{4p}} D(t)^2, \tag{3.5}$$

where $C(u_0, u_1, T)$ denotes the positive quantity given by

$$C(u_0, u_1, T) = \frac{4(p+1)}{4p+N}(C+1) + (C\tilde{C}_1)^{\frac{4(p+1)}{4p+N}}(A_pT)^{\frac{4}{4p+N}} I^{\frac{2N}{4p+N}}.$$

Now applying Lemma 2.2 to the above inequality (3.5), we obtain the estimate

$$E(t) \le \left(E(0)^{-\frac{N}{4p}} + \frac{N}{4p}C(u_0, u_1, T)^{-\left(1+\frac{N}{4p}\right)}(t-T)^+\right)^{-\frac{4p}{N}}$$

for $4 > N$.

Case (ii): $N \ge 4$. By the assumption on $a(x)$ and the generalized Hölder inequality, we have

$$\begin{aligned} \int_t^{t+T}\int_{\tilde{\omega}} |u_t|^2 dxds &= \int_t^{t+T}\int_{\tilde{\omega}} a(x)^r a(x)^{-r}|u_t|^{2r}|u_t|^{2-2r} dxds \\ &\le \left(\int_t^{t+T}\int_{\tilde{\omega}} (a(x)|u_t|^2)^{rp_1} dxds\right)^{\frac{1}{p_1}} \left(\int_t^{t+T}\int_{\tilde{\omega}} a(x)^{-rp_2} dxds\right)^{\frac{1}{p_2}} \left(\int_t^{t+T}\int_{\tilde{\omega}} |u_t|^{(2-2r)p_3} dxds\right)^{\frac{1}{p_3}} \\ &= \left(\int_t^{t+T}\int_{\tilde{\omega}} a(x)|u_t|^2 dxds\right)^{r} \left(\int_t^{t+T}\int_{\tilde{\omega}} a(x)^{-p} dxds\right)^{\frac{r}{p}} \left(\int_t^{t+T} \|u_t\|^{\frac{2p+4}{p}}_{L^{\frac{2p+4}{p}}(\tilde{\omega})} ds\right)^{\frac{p}{3p+2}}, \end{aligned} \tag{3.6}$$

where we use $rp_1 = 1,\ rp_2 = p,\ (2-2r)p_3 = \frac{2p+4}{p}$ and $r = \frac{2p}{3p+2}$. Using the Gagliardo–Nirenberg inequality (see Lemma 2.1) and $p > 0$ satisfying $N - 4 \le 2p$ in the last term of the right-hand side of (3.6) yields

$$\|u_t\|_{L^{\frac{2p+4}{p}}(\tilde{\omega})} \le C_2\|u_t\|^{\frac{N}{2(p+2)}}_{H^2(\Omega)}\|u_t\|^{\frac{2(p+2)-N}{2(p+2)}}_{L^2(\Omega)} \le \tilde{C}_2 I^{\frac{N}{2(p+2)}} E(t)^{\frac{2(p+2)-N}{4(p+2)}}. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned} \int_t^{t+T}\int_{\tilde{\omega}} |u_t|^2 dxds &\le A_p^{\frac{2}{3p+2}} T^{\frac{2}{3p+2}} D(t)^{\frac{4p}{3p+2}} \sup_{t\le s\le t+T} \|u_t\|^{\frac{2p+4}{3p+2}}_{L^{\frac{2p+4}{p}}(\tilde{\omega})} \\ &\le \tilde{C}_2^{\frac{2p+4}{3p+2}} A_p^{\frac{2}{3p+2}} T^{\frac{p+2}{3p+2}} I^{\frac{N}{3p+2}} D(t)^{\frac{4p}{3p+2}} E(t)^{\frac{2(p+2)-N}{2(3p+2)}}. \end{aligned} \tag{3.8}$$

Thus we have from identity (3.1), Proposition 3.1, (3.8) and Young's inequality,

$$\begin{aligned} E(t) &\le (C+1)D(t)^2 + C_3 D(t)^{\frac{4p}{3p+2}} E(t)^{\frac{2(p+2)-N}{2(3p+2)}} \\ &\le (C+1)D(t)^2 + \frac{4p+N}{2(3p+2)}\left(C_3 D(t)^{\frac{4p}{3p+2}}\right)^{\frac{2(3p+2)}{4p+N}} + \frac{2(p+2)-N}{2(3p+2)}E(t), \end{aligned} \tag{3.9}$$

where $C_3 = C\tilde{C}_2^{\frac{2p+4}{3p+2}} A_p^{\frac{2}{3p+2}} T^{\frac{p+2}{3p+2}} I^{\frac{N}{3p+2}}$. This is modified by

$$E(t)^{1+\frac{N}{4p}} \le \tilde{C}(u_0, u_1, T)^{\frac{4p+N}{4p}} D(t)^2, \tag{3.10}$$

where the positive constant $\tilde{C}(u_0, u_1, T)$ is given by

$$\tilde{C}(u_0, u_1, T) = \frac{2(3p+2)}{4p+N}(C+1) + C_3^{\frac{2(3p+2)}{4p+N}}.$$

Applying Lemma 2.2 to (3.10), we obtain the decay estimate

$$E(t) \le \left(E(0)^{-\frac{N}{4p}} + \frac{N}{4p}\tilde{C}(u_0, u_1, T)^{-\left(1+\frac{N}{4p}\right)}(t-T)^+\right)^{-\frac{4p}{N}}$$

for $4 \le N$.

References

[1] C. Bardos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary, SIAM J. Control Optim. 30 (1992) 1024–1065.

[2] M.M. Cavalcanti, V.N. Domingos Cavalcanti, R. Fukuoka, J.A. Soriano, Asymptotic stability of the wave equation on compact surfaces and locally distributed damping—a sharp result, Trans. Amer. Math. Soc. 361 (9) (2009) 4561–4580.

[3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, R. Fukuoka, J.A. Soriano, Asymptotic stability of the wave equation on compact manifolds and locally distributed damping: A sharp result, Arch. Rational. Mech. Anal., doi:10.1007/s00205-009-0284-z.

[4] C.M. Dafermos, Asymptotic behavior of solutions of evolution equations, in: M.G. Crandall (Ed.), Nonlinear Evolution Equations, Funkcial. Ekva. 38 (1995), 545–568.

[5] A. Haraux, Stabilization of trajectories for some weakly damped hyperbolic equations, J. Differential Equations 59 (1985) 145–154.

[6] Y.H. Kang, M.J. Lee, I.H. Jung, Sharp energy decay estimates for the wave equation with a local degenerate dissipation, Comput. Math. Appl. 57 (2009) 21–27.

[7] P. Martinez, Decay of solutions of the wave equation with a local highly degenerate dissipation, Asymptot. Anal. 19 (1999) 1–17.

[8] M. Nakao, Decay of solutions of the wave equation with a local degenerate dissipation, Israel J. Math. 95 (1996) 25–42.

[9] M. Nakao, Global existence of smooth solutions to the initial boundary value problem for the quasi-linear wave equation with a localized degenerate dissipation, Nonlinear Anal. 39 (2000) 187–205.

[10] J. Rauch, M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains, Indiana Univ. Math. J. 24 (1974) 79–86.

[11] L.R. Tcheugoué Tébou, On the decay estimates for the wave equation with a local degenerate or nondegenerate dissipation, Port. Math. 55 (1998) 293–306.

[12] L.R. Tcheugoué Tébou, Stabilization of the wave equation with localized nonlinear damping, J. Differential Equations 145 (1998) 502–524.

[13] E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, Comm. Partial Differential Equations 15 (1990) 205–235.

[14] J.L. Lions, Contrôlabilité exacte, perturbations et stabilization de systèmes distribués, Tome 1, Paris, Masson, 1988.

[15] M.M. Cavalcanti, Existence and uniform decay for the Euler–Bernoulli viscoelastic equation with nonlocal boundary dissipation, Discrete Contin. Dyn. Syst. 8 (3) (2002) 675–695.

[16] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation, Commun. Contemp. Math. 6 (5) (2004) 705–731.

[17] R.C. Charao, E. Bisognin, V. Bisognin, A.F. Pazoto, Asymptotic behavior of a Bernoulli–Euler type equation with nonlinear localized damping, Progr. Nonlinear Differential Equations Appli. 66 (2005) 67–91.

[18] J.U. Kim, Exact semi-internal controllability of an Euler–Bernoulli equations, SIAM J. Control Optim. 30 (1992) 1001–1023.

[19] M. Tucsnak, Semi-internal stabilization for a non-linear Bernoulli–Euler equation, Math. Methods Appl. Sci. 19 (1) (1996) 897–907.